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# A NEW PROOF OF THE COMPETITIVE EXCLUSION PRINCIPLE IN THE CHEMOSTAT

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Abstract. We give an new proof of the well-known competitive exclusion principle in the chemostat model with $n$ species competing for a single resource, for any set of increasing growth functions. The proof is constructed by induction on the number of the species, after being ordered. It uses elementary analysis and comparisons of solutions of ordinary differential equations.

Keywords. Chemostat model, competitive exclusion, global stability, comparison principle, induction.

Mathematics Subject Classification. 34K20, 92D25.

## 1. Introduction

The Competitive Exclusion Principle (CEP) has a long history in the scientific literature. Since the thirties, the Russian botanist Gause conducted experiments on the growth of yeasts and paramecia in mixed cultures, and reported that the most competitive species systematically eliminated the other [5]. In his book "The struggle for existence", he showed that competitive exclusion had indeed a more universal scope: two similar living organisms evolving in the same environment and competing for a shared resource cannot coexist for ever, one of them having always a slight advantage over the other one, or being more adapted to the ecosystem [6]. In the 1960s, this statement has become quite popular in ecology but also in economics: the CEP applies to many kinds of ecosystems, and not only for microorganisms, since there are consumers and resources [7]. It is also commonly taught as "Gause's law" in theoretical ecology.

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However, it was not until the 1970s that the first statement of a mathematical theorem was found in the literature, along with its proof [10], for the chemostat model. It refers to the mathematical result that establishes conditions under which almost all solutions converge toward a steady-state of the system having at most one species. The chemostat model is widely used in microbiology and ecology as a mathematical representation of growth of micro-organisms in ecosystems that are continuously fed with nutrients. Several textbooks on the mathematical analysis of this model with one and more species are available [17, 1, 8]. The chemostat model can be also considered as a quite general resource/consumers model [12]. The CEP has also a long history in the literature of bio-mathematics. Hsu, Hubbell and Waltman have proposed a first proof in 1977 for Monod's functions as particular growth rates [10]. Hsu generalized this result in 1978 for different removal rates [9]. These two contributions use explicit Lyapunov functions to demonstrate the overall convergence. In 1980, Armstrong and McGehee have given a simple proof for any monotonic growth functions but for particular initial conditions belonging to an invariant set [2]. In 1985, Butler and Wolkowicz proposed a proof for any monotonic growth function [3]. One of the difficulties to prove the global stability originates from the fact that the graphs of any growth function can intersect one another at several points. During the transients a species could dominate the competition without being the final winner of the competition on the long run. Finally, it was in 1992 that Wolkowicz and Lu proposed a proof, based on a Lyapunov function, for growth functions more general than Monod functions (but under additional technical assumptions) and different removal rates [18]. This result has been later extended or complemented [11, 14, 13, 16, 15]. However, the proof of global stability for any monotonic growth functions and removal rates remains today an open mathematical problem [4].

In the present paper, we propose a new proof of the CEP for any monotonic growth functions but under identical removal rates. The existing proofs rely on relatively sophisticated tools, such as $\omega$-limit sets [3] or Lyapunov functions [10, 18] and LaSalle Invariance Principle. We show that it is possible to obtain a proof with elementary analysis, based on single comparisons of solutions of ordinary differential equations. While species are sorted in ascending break even concentrations, the key of the proof relies on the observation of the time evolution of the proportions $r_i$ of the concentration of species $i$ over the one of species 1. Whatever is the initial condition and how the transients could exhibit an alternation of dominance among species or not, there always exists a finite time at the end of which the proportion $r_n$ is decreasing exponentially for any future time. We show that this property is due to the level of the resource that reaches in finite time an interval which is unfavorable for the $n$-th species, and belongs to this interval for ever. We show that these two properties hold for the other species by induction on the index set $\{n, n-1, \cdots, 2\}$. This proves that the only winner of the competition is the first species.

## 2. Competitive exclusion principle for the chemostat

The classical chemostat model for $N \in \mathbb{N}^*$ species competing for a single resource is given by the system of differential equations

$$\begin{cases} \dot{s} = & D(S_{\text{in}} - s) - \displaystyle\sum_{i=1}^{N} \mu_i(s) x_i \\ \dot{x}_i = & \mu_i(s) x_i - D x_i \quad (1 \leq i \leq N) \end{cases}, \tag{1}$$

where the operating parameters $D > 0$, $S_{\text{in}} > 0$ are the removal rate and the incoming density of resource (or input substrate concentration). The variable $s(t)$ denotes the density of resource (or substrate concentration) at time $t$. For $1 \leq i \leq N$, $x_i(t)$ represents the density of the $i$-th species and $\mu_i(\cdot)$ is the specific growth rate function of species $i$. In this writing we have assumed, without any loss of generality, that the yield coefficients of resource $s$ transformed in $x_i$ are all identically equal to 1. In microbial ecology, the growth function $\mu_i$ often takes the form of a Monod's function $\mu(s) = \mu_{\max} \dfrac{s}{k+s}$, but we consider here more general ecosystems without particularizing the expression of the growth function. We make the following assumption.

**Assumption 1** (Growth function)**.** *For each $1 \leq i \leq N$, we assume that*

(1) $\mu_i \in C^1(\mathbb{R}^+)$.
(2) $\mu_i(0) = 0$.
(3) $\mu_i(\cdot)$ *is an increasing of function of* $s$.

One can straightforwardly check that the positive orthant of $\mathbb{R}^{N+1}$ is invariant by the dynamics (1).

As often considered in the literature, we associate to each growth function the *break-even concentration* defined as follows.

**Definition 1** (Break-even concentration)**.** *Under Assumption 1, for a given number $D > 0$, the break-even concentration $\lambda_i = \lambda_i(D)$ for the $i$-th species is defined as the unique solution of the equation $\mu_i(s) = D$, when it exists. When there is no solution to this equation, we set $\lambda_i = \infty$.*

**Assumption 2.** *Species have distinct break-even concentrations, and without loss of generality are enumerated by indices such that*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N. \tag{2}$$

In Figure 1, we have represented several growth functions and the removal rate (in dashed line). We shall see further how the particular cases when some numbers $\lambda_i$ are identical can be tackled, packing the corresponding species (see Section 3.5 below). We first do not consider these non generic situations for sake of simplicity of the presentation.

We recall the statement of the Competitive Exclusion Principle.

**Proposition 1** (Competitive exclusion principle)**.** *Assume one has $\lambda_1 < S_{\text{in}}$ with Assumptions 1 and 2 fulfilled. For any non-negative initial condition with $x_1(0) > 0$, the solution of the system* (1) *converges to the equilibrium point $(\lambda_1, S_{\text{in}} - \lambda_1, 0, \ldots, 0)$.*

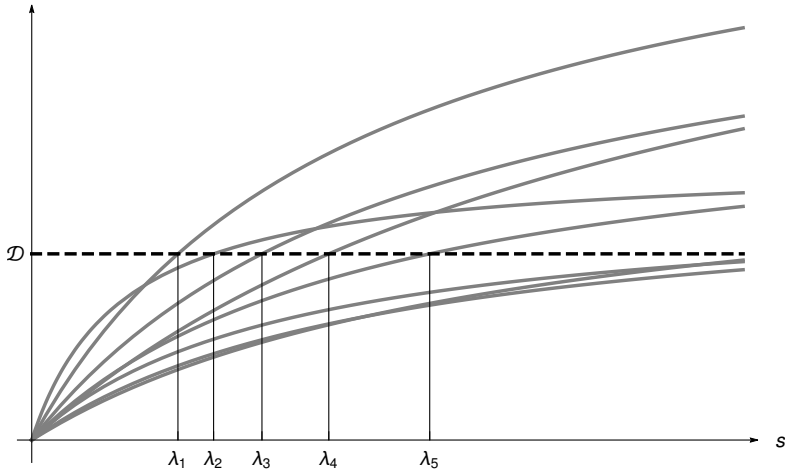



FIGURE 1. Growth functions and their break-even concentrations.

## 3. Proof of Proposition 1

The proof is based on a backward inductive argument: we show that the proportion, with respect of the total biomass, of each species goes to 0 when $t$ tends to $+\infty$ excepted for the species with minimal break-even concentration.

3.1. **Change of coordinates.** We introduce the *total biomass* $b$ and the *proportion's vector* $p := (p_i)_{1\leq i\leq N}$ where:

$$b := \sum_{i=1}^{N} x_i \quad \text{and} \quad p_i := \frac{x_i}{b}. \tag{3}$$

Additionally, we define the function $\bar{\mu}(s,p) := \sum_{i=1}^{N} p_i\mu_i(s)$. In those new variables, one can easily check that the system (1) writes:

$$\left\{\begin{array}{lcll} \dot{s} & = & D(S_{\text{in}} - s) - \bar{\mu}(s,p)b & \\ \dot{b} & = & \bar{\mu}(s,p)b - Db & \\ \dot{p}_i & = & p_i(\mu_i(s) - \bar{\mu}(s,p)) & (1 \leq i \leq n). \end{array}\right. \tag{4}$$

3.2. **Non extinction of the total biomass.** We first give a necessary and sufficient condition for the persistence of the total biomass, of interest in itself.

**Lemma 1.** *Consider that Assumptions 1 and 2 are fulfilled.*

(1) *Whatever is the initial condition of* (1) *, the solution verifies*

$$\lim_{t\to+\infty} b(t) + s(t) = S_{in}.$$

(2) *Any species $j$ with $\lambda_j \geq S_{\text{in}}$ satisfies $\lim_{t\to+\infty} x_j(t) = 0$, whatever is the initial condition of* (1)*.*

(3) *When $\lambda_1 < S_{\text{in}}$, for any non-negative initial condition such that there exists $i \in \{1, \cdots, n\}$ with $x_i(0) > 0$ and $\lambda_i < S_{\text{in}}$, the variable $b(t)$ is bounded from below by a positive number for any $t > 0$.*

*Proof.* Consider the *total mass* $m := b + s$ of the system. Then $m$ solves the linear differential equation $\dot{m} = D(S_{\text{in}} - m)$, which posses an unique equilibrium $m^\star = S_{\text{in}}$, that is moreover globally asymptotically stable.

Consider first species $j$ such that $\lambda_j \geq S_{\text{in}}$ (if it exists). Fix $\epsilon > 0$. Then $\eta = D - \mu_j(S_{\text{in}} - \epsilon/2)$ is a positive number. As $m(\cdot)$ converges to $S_{\text{in}}$, there exits $T > 0$ such that $s(t) + x_j(t) \leq m(t) < S_{\text{in}} + \epsilon/2$ for any $t > T$. This implies that the dynamics of species $j$ satisfies

$$\dot{x}_j(t) \leq \left(\mu_j(S_{\text{in}} - x_j(t) + \epsilon/2) - D\right) x_j(t) =: \phi_j(x_j(t))$$

for any $t > T$. The function $\phi_j$ has the property

$$x_j \geq \epsilon \;\Rightarrow\; \phi_j(x_j) \leq \left(\mu_j(S_{\text{in}} - \epsilon/2) - D\right) x_j \leq -\eta\epsilon < 0.$$

Therefore, the variable $x_j(t)$ exits the domain $\{x_j \geq \epsilon\}$ in finite time and stays outside for any future time, i.e. there exists $T' > T$ such that $x_j(t) < \epsilon$ for any $t > T'$. This statement is obtained for any arbitrary $\epsilon$, which proves the convergence of the $x_j(\cdot)$ towards 0.

Let $k$ be the maximal index such that $\lambda_k < S_{\text{in}}$ (which exists by Assumption 2) and denote $b_0 := x_1 + \cdots + x_k$. For any $i \leq k$, notice that one has

$$\mu_i(s)x_i \geq \min_{1 \leq j \leq k} \mu_j(s)x_i$$

for any $s$ and $x_i$, and by simple addition,

$$\sum_{i=1}^{k} \mu_i(s)x_i \geq \min_{1 \leq j \leq k} \mu_j(s)\Big(\sum_{i=1}^{k} x_i\Big) = \min_{1 \leq j \leq k} \mu_j(s)b_0. \tag{5}$$

The dynamics of $b_0$ writes

$$\dot{b}_0 = \sum_{i=1}^{k} \dot{x}_i = \sum_{i=1}^{k} \mu_i(s)x_i - D\sum_{i=1}^{k} x_i$$

and by inequality (5), one has $\dot{b}_0(t) \geq \Psi(t, b_0(t))$ for any $t > 0$, where $\Psi$ is defined as follows

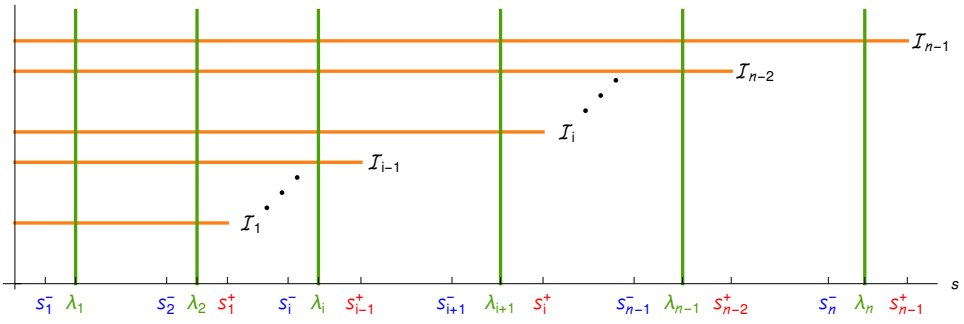
$$\Psi(t, b_0) := \big(\min_{1 \leq i \leq k} \mu_i(s(t)) - D\big)\, b_0. \tag{6}$$

As the inequality $\mu_i(S_{\text{in}}) \geq \mu_1(S_{\text{in}}) > D$ is fulfilled for any $i \leq k$, and the functions $\mu_i$ are continuous, there exists $\epsilon > 0$ and $\eta > 0$ such that

$$\min_{1 \leq i \leq k} \mu_i(\sigma) - D > \epsilon \text{ for any } \sigma > S_{\text{in}} - \eta. \tag{7}$$

Since $m(t)$ converges to $S_{\text{in}}$ and $x_j(t)$ converges to 0 for any $j > k$, there is a time $T^\star \geq 0$ such that

$$s(t) = m(t) - b_0(t) - \sum_{j \geq k} x_j(t) > S_{\text{in}} - b_0(t) - \frac{\eta}{2} \text{ for any } t > T^\star.$$




Then, by inequality (7), the function $\Psi$ defined in (6) fulfills the following property.

$$\left\{t > T^\star, b_0 < \tfrac{\eta}{2}\right\} \Rightarrow \Psi(t, b_0) \geq \big(\min_{1\leq i\leq k} \mu_i(S_{\text{in}} - \eta) - D\big) b_0 \geq \epsilon\, b_0.$$

As one has $b_0(0) > 0$ by hypothesis, $b_0(t)$ is strictly positive for any $t > T^\star$ and it follows that $b_0$ can not stay or enter into the interval $[0, \frac{\eta}{2}]$ for times larger than $T^\star$. Then the total biomass $b(t) \geq b_0(t)$ is bounded from below by $\frac{\eta}{2}$ in finite time. □

3.3. **Frame on the substrate's dynamics.** By Assumption 2, growth functions are ordered on each interval $[\lambda_i, \lambda_{i+1}]$ (see Figure 1) i.e. for all $i \in \{1 \cdots n-1\}$, one has

$$\mu_i(s) > \mu_j(s), \quad \forall j > i, \quad \forall s \in [\lambda_i, \lambda_{i+1}].$$

By continuity of $\mu_i(\cdot)$, there are numbers $\nu > 0$, $s_i^- < \lambda_i$ and $s_i^+ > \lambda_{i+1}$ such that

$$\mu_i(s) > \mu_j(s) + \nu, \quad \forall s \in [s_i^-, s_i^+], \quad \forall j > i. \tag{8}$$

In particular, one has

$$\mu_1(s) > \mu_i(s) + \nu, \quad \forall s \in [s_1^-, s_i^+], \quad i > 1. \tag{9}$$

By monotonicity of $\mu_i$, we have $\mu_i(s_i^-) < D$ and $\mu_i(s_i^+) > D$. Moreover, one has

$$\begin{aligned} j > i \quad &\Rightarrow \quad \mu_j(s_i^-) < \mu_i(s_i^-) < D \text{ (by inequality (8))} \\ j < i \quad &\Rightarrow \quad \mu_j(s_i^+) > \mu_j(\lambda_i(D)) > \mu_j(\lambda_j(D)) = D. \end{aligned}$$

Therefore, the numbers

$$\left\{\begin{array}{rcl} \gamma^- & := & D - \max\limits_{1\leq i\leq n-1} \max\limits_{j>i} \mu_j(s_i^-) \\ \gamma^+ & := & \min\limits_{1\leq i\leq n-1} \min\limits_{j<i} \mu_j(s_i^+) - D \end{array}\right. \tag{10}$$

are positive, and we define the constants

$$D^- := D - \frac{\gamma^-}{2}, \quad D^+ := D + \frac{\gamma^+}{2}\,. \tag{11}$$

**Lemma 2.** *Assume one has $\lambda_1 < S_{\text{in}}$ with Assumptions 1 and 2. For any non-negative initial condition with $x_1(0) > 0$, there exists a number $T > 0$ such that $\dot{s}(t) \in [\Phi^-(t, s(t)), \Phi^+(t, s(t))]$ for any $t > T$, where*

$$\left\{\begin{array}{rcl} \Phi^-(t,s) & := & [D^- - \bar{\mu}(s, p(t))]\, b(t) \\ \Phi^+(t,s) & := & [D^+ - \bar{\mu}(s, p(t))]\, b(t). \end{array}\right. \tag{12}$$

*Proof.* From Lemma 1, we know that $b(\cdot)$ is bounded from below by a positive number. Consider the function

$$z := \frac{S_{in} - s}{b} = 1 + \frac{S_{in} - m}{b}.$$

Since $m(\cdot)$ converges to $S_{in}$ and $b(\cdot)$ is bounded form below, $z(t)$ converges to 1 when $t$ tends to $+\infty$. Furthermore, for $t > T$ with $T$ large enough, one has $Dz(t) \in [D^-, D^+]$. Remark that the substrate dynamics can be written as follows

$$\dot{s} = \Phi(t, s) := b(t) \left[ Dz(t) - \bar{\mu}(s, p(t)) \right] ,$$

and one then obtains the inequalities $\Phi^-(t, s) \leq \Phi(t, s) \leq \Phi^+(t, s)$ for any $t > T$. □

### 3.4. **Recursive extinction of the species.**

**Proposition 2.** *Assume one has $\lambda_1 < S_{\rm in}$ with Assumptions 1 and 2. Consider a solution of system (1) with $x_1(0) > 0$. Then the property*

$$(\mathcal{P}_i): \quad \begin{cases} i. & \text{There exists } T_i > T \text{ s.t. } s(t) \in I_i := [s_1^-, s_i^+],\ \forall t > T_i \\ ii. & \lim\limits_{t \to +\infty} p_j(t) = 0,\ \forall j > i. \end{cases}$$

*holds for all $i \in \{1, \ldots, n-1\}$.*

*Proof.* As $x_1(0) > 0$, the variable $x_1$ stays positive for any time and one can consider variables $r_i := \frac{x_i}{x_1}$ for $i \in \{2, \cdots, n\}$, whose dynamics is

$$\dot{r}_i = [\mu_i(s(t)) - \mu_1(s(t))] r_i. \tag{13}$$

By monotonicity of functions $\mu_i(\cdot)$ and definitions of (10) and (11), we have

$$\begin{aligned} s < s_1^- \Rightarrow \max_i \mu_i(s) < \max_i \mu_i(s_1^-) &= \mu_1(s_1^-) \\ &\leq \max_i \mu_i(s_i^-) = D^- - \frac{\gamma^-}{2}. \end{aligned} \tag{14}$$

Similarly, we can write

$$\begin{aligned} s > s_{n-1}^+ \Rightarrow \min_i \mu_i(s) > \min_i \mu_i(s_{n-1}^+) &\geq \min_i \mu_i(s_i^+) \\ &= \min_{1 \leq i \leq n-1} \mu_i(s_i^+) = D^+ + \frac{\gamma^+}{2}. \end{aligned} \tag{15}$$

Consider a number $T > 0$ given by Lemma 2. From Lemma 1, there exists a number $\eta > 0$ such that $b(t) \geq \frac{\eta}{2}$ for any $t > T$. It then follows from (14) and (15) that the functions $\Phi^{\pm}$ defined in (12) fulfill the following inequalities.

$$\begin{aligned} s < s_1^-,\ t > T \quad &\Rightarrow \quad \Phi^-(t, s) \geq b(t) \left( D^- - \max_i \mu_i(s) \right) > \frac{\eta \gamma^-}{4} > 0 \\ s > s_{n-1}^+,\ t > T \quad &\Rightarrow \quad \Phi^+(t, s) \leq b(t) \left( D^+ - \min_i \mu_i(s) \right) < -\frac{\eta \gamma^+}{4} < 0. \end{aligned}$$

Therefore the variable $s$ enters the interval $I_{n-1}$ in a finite time $T_{n-1} > T$ and belongs to it for any future time. Furthermore, the inequality (9) for $i = 1$ ensures to have $\mu_n(s) - \mu_1(s) < -\nu$ for any $s$ in the interval $I_{n-1}$, and then the dynamics of $r_n$ satisfies $\dot{r}_n \leq -\nu r_n$ for $t > T_{n-1}$. Thus $r_n$ converges to 0, and $x_n$ converges as well. Property $\mathcal{P}_{n-1}$ is then satisfied.

Assume that $\mathcal{P}_i$ is satisfied for an index $i \in \{2, \cdots, n-1\}$ and let us show that $\mathcal{P}_{i-1}$ is fulfilled. Since the functions $p_j$ converge to 0 for all $j > i$, there is $T' > T_i$ such that

$$\sum_{j>i} p_j(t) < \varepsilon := \frac{\gamma^+/4}{D^+ + \gamma^+/2}, \quad \forall t > T'.$$

Then, for $s > s^+_{i-1}$ and $t > T'$, the following inequality holds

$$\begin{aligned}\sum_{j\le i} \mu_j(s)p_j(t) &\geq \mu(s, p(t)) \\ &\geq \min_{j\le i} \mu_j(s^+_{i-1})(1-\varepsilon) \geq \min_{j\le i} \mu_j(s^+_j)(1-\varepsilon) \\ &\geq (D^+ + \tfrac{\gamma^+}{2})(1-\varepsilon) = D^+ + \frac{\gamma^+}{4}\end{aligned}$$

which provides the property of the function $\Phi^+$:

$$s > s^+_{i-1},\ t > T' \ \Rightarrow\ \Phi^+(t,s) \leq -\frac{\eta\,\gamma^+}{8} < 0.$$

Thus, in a finite time $T_{i-1} > T'$, $s$ enters into the interval $I_{i-1}$ and stays inside it for any future time. Furthermore, equation (9) leads to the inequality $\dot{r}_i \leq -\nu r_i$ for $t > T_{i-1}$, which shows that $x_i$ converges to zero. Property $\mathcal{P}_{i-1}$ is then satisfied. □

3.5. **The case of identical break-even concentration.** We relax Assumption 2 allowing some $\lambda_i$ with $i > 1$ to be identical and show that Proposition 1 is also satisfied.

If there exist $i$ and $\ell > 1$ such that $\lambda_1 < \lambda_{i-\ell}(D) = \cdots = \lambda_i(D) < S_{\text{in}}$, at step $i$ in the recursive proof, we replace species $i$ by the sum of species $i-\ell, \ldots, i$ and show property $\mathcal{P}_i$, that is $\lim_{t\to+\infty} p_j(t) = 0$ for all $j > i-\ell+1$.

This can be done considering $\overline{r}_i := \sum_{j=i-\ell}^{i} r_j$ instead of $r_i$, and remark that we have also $s^+_{i-1} = \cdots = s^+_{i-1-\ell}$. Then, one has

$$\dot{\overline{r}}_i = \left[\sum_{j=i-\ell}^{i} \alpha_j(t)\mu_j(s(t)) - \mu_1(s(t))\right] r_i$$

where $\alpha_j(t) = \dfrac{r_j(t)}{\overline{r}_i(t)} > 0$ with $\sum_{j=i-1}^{i} \alpha_j(t) = 1$. By (9), we have

$$\sum_{j=i-1}^{i} \alpha_j(t)\mu_j(s) < \mu_1(s) - \nu, \quad \forall s \in [s^-_1, s^+_i]$$

an then $\dot{\overline{r}}_i(t) < -\eta\overline{r}_i(t)$. Thus $\dot{\overline{r}}_i \leq \eta\,\overline{r}_i$ for $t > T$ which shows that $\overline{r}_i$ converges to zero as well as every $x_j$ with $i \leq j \leq i+\ell$. The rest of the proof is identical.

3.6. **Conclusion.** Now that we know that the variables $x_j$ converge to 0 for any $j > 1$, we can easily show that $s$ tends to $\lambda_1$ as follows.

By Lemma 1 and Proposition 2, $b(t) + s(t)$ tends to $S_{\text{in}}$ and the variables $b$ and $s$ are bounded from below by positive numbers. Thus there exists numbers $\zeta > 0$ and $T_1 > 0$ such that $s(t) \in [\zeta, S_{\text{in}} - \zeta]$ for any $t > T_1$, and accordingly to Proposition 2, we can require to have $\lambda_1 \in (\zeta, S_{\text{in}} - \zeta)$. We then consider the variable

$$r(t) = \frac{\bar{\mu}(s(t), p(t))b(t)}{\mu_1(s(t))(S_{\text{in}} - s(t))}, \quad t > T_1$$

which tends to 1 when $t$ tends to $+\infty$, and write the dynamics of $s$ as

$$\dot{s} = \Gamma(t, s) := (S_{\text{in}} - s)(D - \mu_1(s)r(t)) \tag{16}$$

Take any $\epsilon > 0$ sufficiently small to have $[\lambda_1 - \epsilon, \lambda_1 + \epsilon] \subset [\zeta, S_{\text{in}} - \zeta]$. The function $\mu_1(\cdot)$ being assumed to be of class $C^1(\mathbb{R}^+)$, increasing and with $\mu_1(\lambda_1) = D$, there exists $T_2 > T_1$ and $\eta > 0$ such that

$$\begin{array}{l} t > T_2,\ s > \lambda_1 + \epsilon \Rightarrow \mu_1(s)r(t) > D + \eta \\ t > T_2,\ s < \lambda_1 - \epsilon \Rightarrow \mu_1(s)r(t) < D - \eta \end{array}$$

Then, the function $\Gamma$ fulfills the following properties

$$\begin{array}{l} t > T_2,\ s \in [\lambda_1 + \epsilon, S_{\text{in}} - \zeta] \Rightarrow \Gamma(t, s) < -\eta\zeta < 0 \\ t > T_2,\ s \in [\zeta, \lambda_1 - \epsilon] \Rightarrow \Gamma(t, s) > \eta(S_{\text{in}} - \zeta) > 0 \end{array}$$

which allows to conclude that the variable $s$ converges to the interval $[\lambda_1 - \epsilon, \lambda_1 + \epsilon]$, and this can be obtained for any arbitrarily small $\epsilon > 0$.

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